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A CONJECTURE OF AX AND DEGENERATIONS OF FANO VARIETIES

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A field k is called C_1 if every homogeneous form $f(x_0,\dots,x_n)\in k[x_0,\dots,x_n]$ of degree $\leq n$ has a nontrivial zero. Examples of C_1 fields are finite fields (Chevalley) and function fields of curves over an algebraically closed field (Tsen).

A field is called *PAC* (pseudo algebraically closed) if every geometrically integral k-variety has a k-point. An k-variety X is called geometrically integral (or absolutely irreducible) if it is still irreducible and reduced as a variety over the algebraic closure $\bar{k}$.) These fields were introduced in [Ax68]; see [FJ05] for an exhaustive and up to date treatment.

The aim of this paper is to prove in characteristic 0 a conjecture of Ax, posed in [Ax68, Problem 3].

Theorem 1. *Every PAC field of characteristic 0 is C_1.*

[Ax68, Thm.D] proves this for fields whose absolute Galois group is abelian.

Following an idea of [DJL83], we deduce (1) from the next result which holds for all fields of characteristic zero:

Theorem 2. *Let k be a field of characteristic 0 and $f_1,\dots,f_s\in k[x_0,\dots,x_n]$ homogeneous polynomials such that $\sum_i \deg f_i\leq n$. Let*

$$X=X(f_1,\dots,f_s):=(f_1=\dots=f_s=0)\subset\mathbb{P}^n_k$$

be the subscheme they define in projective n-space. Then

1. *X contains a geometrically irreducible k-subvariety $Y\subset X$.*
2. *If k is PAC then X has a k-point.*

If k is PAC, then Y has a k-point which is also a k-point of X, thus (2.1) implies (2.2). The $s=1$ case of (2.2) is precisely (1). The more general version proved here is sometimes called property C_1'.

In order to prove (2), we represent (a subscheme of) the scheme $X(f_1,\dots,f_s)$ as a special fiber of a family $Z\to\mathbb{P}^1$ over the projective line whose general fiber is a smooth hypersurface (resp. complete intersection variety). The restrictions on the degree are equivalent to assuming that the canonical class of the general fiber of $Z\to\mathbb{P}^1$ is negative. This approach raises further interesting questions about degenerations of Fano varieties, we discuss these in (11).

It is thus sufficient to prove the following more general result.

Theorem 3. *Let k be a field of characteristic 0, C a smooth k-curve, Z an irreducible, projective k-variety and $g: Z\to C$ a morphism. Assume that general fibers F_{gen} are*

1. *smooth,*
2. *geometrically connected, and*
3. *Fano (that is, $-K_{F_{gen}}$ is ample).*

Then every fiber $g^{-1}(c)$ contains a $k(c)$-subvariety which is geometrically irreducible. In particular, if every $k(c)$-irreducible component of $g^{-1}(c)$ is smooth (or normal), then $g^{-1}(c)$ contains a $k(c)$-irreducible component which is geometrically irreducible.

Let us start by proving that (3) $\Rightarrow$ (2).

Take homogeneous polynomials $g_1, \dots, g_s \in k[x_0, \dots, x_n]$ such that $\deg g_i = \deg f_i$ and $(g_1 = \cdots = g_s = 0) \subset \mathbb{P}^n_k$ is a smooth complete intersection of dimension $n - s$.

Let $Z_1 \subset \mathbb{P}^n_k \times \mathbb{P}^1_k$ be defined by the equations

$$(uf_1 + vg_1 = \cdots = uf_s + vg_s = 0) \subset \mathbb{P}^n_k \times \mathbb{P}^1_k,$$

where $(u : v)$ are the coordinates on the projective line $\mathbb{P}^1$. Let $Z \subset Z_1$ be the unique irreducible component which dominates $C := \mathbb{P}^1$ with projection $f : Z \to C$.

Observe that the fiber over $(0 : 1)$ is the smooth complete intersection variety $(g_1 = \cdots = g_s = 0)$, thus all but finitely many fibers are smooth complete intersection varieties. The canonical class of these fibers is the restriction of $\mathcal{O}_{\mathbb{P}^n}(\sum \deg f_i - n - 1)$, which is negative by assumption. Thus by (3), the fiber of $f : Z \to C$ over $(1 : 0)$ contains a geometrically irreducible k-subvariety Y which is also a k-subvariety of X. $\square$

The proof of (3) proceeds in two steps. First we use resolution of singularities to reduce to the case $f : Y \to C$ where every fiber is a normal crossing divisor. By this I mean that for every $c \in C$, every $k(c)$-irreducible component of $f^{-1}(c)$ is smooth. We can also achieve that there is an ample divisor of the form $-mK_Y + F$ where $m > 0$ and F is contained in the union of some fibers of f.

Then we apply a variant of the Kollár–Shokurov Connectedness Theorem [Kol92, 17.4] to a carefully chosen auxiliary $\mathbb{Q}$-divisor D to prove that every fiber contains a geometrically irreducible component. One of the assumptions of [Kol92, 17.4] is, however, not satisfied in our case, but this is compensated by other special features of the current situation. Thus I go through the whole proof in (6) following a quick introduction to $\mathbb{Q}$-divisors.

Definition 4. Let Y be a smooth k-variety. A prime divisor P is an irreducible and reduced codimension 1 subvariety. A $\mathbb{Q}$-divisor is a formal linear combination

$$D = \sum a_i P_i \quad \text{where } a_i \in \mathbb{Q},$$

and the P_i are prime divisors, usually assumed distinct.

D is called a simple normal crossing divisor if the P_i are smooth and intersect each other transversally.

The support of D is $\operatorname{Supp} D := \cup_{i:a_i \neq 0} P_i$. D is called effective if $a_i \geq 0$ for every i. Every $\mathbb{Q}$-divisor can be uniquely written as

$$D = D^+ - D^- \quad \text{where} \quad D^+ := \sum_{i:a_i>0} a_i P_i \quad \text{and} \quad D^- := \sum_{i:a_i<0} (-a_i) P_i$$

are the positive (resp. negative) parts of D. D^+ and D^- are effective and have no common prime divisors. Given $D = \sum a_i P_i$, set

$$D_{\geq 1} := \sum_{i:a_i \geq 1} a_i P_i.$$

A $\mathbb{Q}$-divisor D is called ample if mD is an ample (integral) divisor for some (or all) $m > 0$ such that ma_i is an integer for every i.

Two $\mathbb{Q}$-divisors D_1, D_2 are called $\mathbb{Q}$-linearly equivalent (denoted by $D_1 \sim_{\mathbb{Q}} D_2$) if there is an integer $m > 0$ such that mD_1 and mD_2 are linearly equivalent integral divisors. (Note that even if the D_i are integral divisors, $\mathbb{Q}$-linear equivalence is slightly different from linear equivalence if $\operatorname{Pic}(Y)$ contains torsion classes.)

We use is the following generalization of the Kodaira vanishing theorem. (See [KM98, Sec.2.5] for a relatively short proof and for further references. Note that there is a misprint in the relevant Corollary 2.68. In the last line $\omega_Y \otimes M$ should be $\omega_Y \otimes L$.)

Theorem 5 (Kawamata–Viehweg vanishing)**.** *Let Y be a smooth, projective variety, W any variety and $f : Y \to W$ a morphism. Let M and $\Delta = \sum a_i P_i$ be $\mathbb{Q}$-divisors and L an integral divisor with the following properties:*

(1) *M is ample,*
(2) *Δ is a simple normal crossing divisor and $0 \le a_i < 1$ for every i,*
(3) *$L \sim_{\mathbb{Q}} M + \Delta$.*

Then $R^i f_*(\mathcal{O}_Y(K_Y + L)) = 0$ for $i \ge 1$.

I want to stress that even in the special case when the general fiber of f is a smooth hypersurface (which is all one needs for (1)) the flexibility provided by $\mathbb{Q}$-divisors is crucial.

Now we can prove the key technical result of this paper.

Theorem 6 (Connectedness theorem)**.** *Let Y be a smooth, projective variety, C a smooth curve and $f : Y \to C$ a morphism with geometrically connected fibers. Let $D = \sum a_i P_i$ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y such that*

(1) *$\operatorname{Supp} D$ has simple normal crossings,*
(2) *D is f-vertical (that is, its support is contained in finitely many fibers of f), and*
(3) *$-(K_Y + D)$ is ample.*

Then every fiber of $f : \operatorname{Supp} D_{\ge 1} \to C$ is geometrically connected.

Proof. The conclusion is geometric, thus we may assume that we are over an algebraically closed field.

For $D = D^+ - D^-$, write $D^+ = A + \Delta^+$ and $D^- = B - \Delta^-$ where A, B are effective and integral, Δ^+ and Δ^- are effective and if we write $\Delta^+ + \Delta^- = \sum a_i P_i$ then $0 \le a_i < 1$ for every i. Note that $\operatorname{Supp} A = \operatorname{Supp} D_{\ge 1} \subset \operatorname{Supp} D^+$ and $\operatorname{Supp} B = \operatorname{Supp} D^-$, thus B is f-vertical and A and B have no irreducible components in common.

Consider the exact sequence

$$0 \to \mathcal{O}_Y(B - A) \to \mathcal{O}_Y(B) \to \mathcal{O}_A(B) \to 0,$$

and apply f_* to get

$$f_*\mathcal{O}_Y(B) \to f_*\mathcal{O}_A(B) \to R^1 f_*\mathcal{O}_Y(B - A).$$

Since B is vertical and f has geometrically connected fibers, $f_*\mathcal{O}_Y(B)$ is a torsion free sheaf which is generically isomorphic to $\mathcal{O}_C$. Thus $f_*\mathcal{O}_Y(B)$ is a rank 1 locally free sheaf on C. Observe that

$$\begin{aligned} B - A &\sim_{\mathbb{Q}} D^- + \Delta^- - D^+ + \Delta^+ \sim_{\mathbb{Q}} -D + (\Delta^- + \Delta^+) \\ &\sim_{\mathbb{Q}} K_Y + \bigl(-(K_Y + D)\bigr) + (\Delta^- + \Delta^+), \end{aligned}$$

hence by (5) we conclude that $R^1 f_*\mathcal{O}_Y(B-A)=0$ and so $f_*\mathcal{O}_A(B)$ is the quotient of the rank 1 locally free sheaf $f_*\mathcal{O}_Y(B)$.

B is an effective divisor which has no irreducible components in common with A, thus we have an injection $\mathcal{O}_A \hookrightarrow \mathcal{O}_A(B)$ and so an injection $f_*\mathcal{O}_A \hookrightarrow f_*\mathcal{O}_A(B)$.

Therefore $f_*\mathcal{O}_A$ is the quotient of a subsheaf M of the rank 1 locally free sheaf $f_*\mathcal{O}_Y(B)$. Since C is a smooth curve, its local rings $\mathcal{O}_{c,C}$ are principal ideal domains, hence M itself is a rank 1 locally free sheaf.

Finally, set $A(c) := A \cap f^{-1}(c)$ and let $A(c)_1, \dots, A(c)_m$ be its connected components. We have surjections

$$\mathcal{O}_{c,C} \cong M_c \twoheadrightarrow f_*\mathcal{O}_{A(c)} \cong H^0(A(c), \mathcal{O}_{A(c)}) \cong \sum_{i=1}^m H^0(A(c)_i, \mathcal{O}_{A(c)_i}) \twoheadrightarrow k^m.$$

This implies that $m=1$, hence $f|_A$ has geometrically connected fibers. $\square$

Corollary 7. *Let Y be a smooth, projective variety, C a smooth curve and $f : Y \to C$ a morphism with geometrically connected fibers. Let $D = \sum a_i P_i$ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y such that*

(1) $\operatorname{Supp} D +$ *(any fiber of f) has simple normal crossings,*
(2) *D is f-vertical,*
(3) *$-(K_Y + D)$ is ample.*

Them, for every $c \in C$, the fiber F_c contains a $k(c)$-irreducible component which is geometrically irreducible.

Proof. Given $c \in C$, let A be any divisor on C which is linearly equivalent to 0 such that $c \in \operatorname{Supp} A$. Let $F_A := f^{-1}(A)$. F_A is linearly equivalent to 0, thus we can add any rational multiple of F_A to D without changing the assumptions of (6). Hence we may assume that

(4) in a neighborhood of F_c, every irreducible component of D has coefficient ≤ 1, and
(5) at least one irreducible component of F_c has coefficient 1 in D.

Let $E \subset F_c$ be such a component. We claim that E is geometrically irreducible.

To see this, let m be the multiplicity of E in F_A and consider $D' = \sum a_i' P_i$ defined by

$$D' := D - \tfrac{\epsilon}{m} F_A + \epsilon E \quad \text{for } 0 < \epsilon \ll 1.$$

This choice assures that in a neighborhood of F_c, E is the only irreducible component of F_c which has coefficient 1 in D'. Furthermore

$$-(K_Y + D') \sim_{\mathbb{Q}} -(K_Y + D) - \epsilon E$$

is still ample for $0 < \epsilon \ll 1$ since ampleness is an open condition by Kleiman's criterion [Kle66]. Thus, in a neighborhood of F_c, $E = D'_{\geq 1}$ and so E is a smooth and geometrically connected variety, hence geometrically irreducible. $\square$

Remark 8. The assumption (7.2) can be relaxed considerably. The proof works without changes if (7.2) is replaced by the following two conditions:

(2.i) $D_{\geq 1}$ is f-vertical, and
(2.ii) $H^0(F_{gen}, \mathcal{O}_{F_{gen}}(\lceil D^- \rceil))$ is 1-dimensional over $k(C)$, where F_{gen} is the generic fiber of f.

The general case is also useful since it can be used to prove that (3) also holds when the generic fiber is a $\mathbb{Q}$-Fano variety

Proof of (3). Choose an embedding of the generic fiber $F_{gen} \hookrightarrow \mathbb{P}^N$ such that $-mK_{F_{gen}} \sim H|_{F_{gen}}$ for some $m \geq 1$ where H is the hyperplane class on $\mathbb{P}^N$. Let B be a very ample divisor on C. Then $\pi_C^*B + \pi_P^*H$ is very ample on $\mathbb{P}^N \times C$ where π_C, π_P are the coordinate projections.

By taking the closure of F_{gen} we get a variety $V \subset \mathbb{P}^N \times C$ with a projection map $\pi : V \to C$ and a birational map $\phi : V \dashrightarrow Z$ which is an isomorphism between $V^0 := \pi^{-1}(C^0)$ and $Z^0 := f^{-1}(C^0)$ for some open $C^0 \subset C$. Moreover, by shrinking C^0 if necessary, we may also assume that $V^0 \to C^0$ is smooth and

$$-mK_V|_{V^0} \sim (\pi_C^*B + \pi_P^*H)|_{V^0}.$$

We can apply Hironaka's resolution theorem to get $h_1 : V_1 \to V$ such that V_1 is smooth and ϕ lifts to a morphism $\phi_1 : V_1 \to Z$.

Let $F^{sing} \subset V_1$ be the union of all singular fibers of $\pi \circ h_1$. Applying Hironaka's resolution theorem again to $F^{sing} \subset V_1$ we get $h_2 : V_2 \to V_1$ and $h : V_2 \to V$ such that every fiber of $\pi \circ h : V_2 \to C$ is a simple normal crossing divisor. That is, for every $c \in C$. every $k(c)$-irreducible component of the reduced fiber $\operatorname{red} F_c$ is smooth.

Throughout these resolutions we do not blow up anything above C^0.

Since $h : V_2 \to V$ is a composite of blow ups of subvarieties, there is an $m_2 > 0$ and an h-exceptional divisor E such that $h^*(m_2(\pi_C^*B + \pi_P^*H)) - E$ is ample on V_2 (cf. [Har77, II.7.10.b and II.7.13]). Dividing by $m \cdot m_2$ we conclude that there is an ample $\mathbb{Q}$-divisor M on V_2 such that

$$-K_{V_2}|_{V^0} \sim_{\mathbb{Q}} M|_{V^0}.$$

Thus there is a $\mathbb{Q}$-divisor D supported in $V_2 \setminus V^0$ such that

$$-(K_{V_2} + D) \sim_{\mathbb{Q}} M.$$

Since the support of D is contained in a union of fibers of $\pi \circ h$, it is a simple normal crossing divisor. Thus D is vertical and the assumptions (7.1–3) hold.

Hence by (7), every fiber of $\pi \circ h : V_2 \to C$ contains a geometrically irreducible component.

Since every fiber of $f : Z \to C$ is dominated by a fiber of $\pi \circ h : V_2 \to C$, we conclude that every fiber of $f : Z \to C$ contains a geometrically irreducible subvariety.

Finally, assume that every $k(c)$-irreducible component of $g^{-1}(c)$ is smooth (or normal). Let $W \subset g^{-1}(c)$ be a geometrically irreducible subvariety and $F \subset g^{-1}(c)$ an irreducible component containing W. Write $F_{\bar{k}} = F_1 + \cdots + F_m$ where the F_i are irreducible over $\bar{k}$. One of the F_i contains $W_{\bar{k}}$, but then so do all the others since the F_i are conjugate over k. Since $F_{\bar{k}}$ is normal, this implies that $m = 1$ and F is geometrically irreducible. $\square$

Theorem (3) naturally raises the following question:

Question 9. Which "natural" classes of schemes $\mathbb{S}$ satisfy the following property

$$\begin{array}{c}\text{For every field } k \text{ and for every } k\text{-scheme } S \in \mathbb{S},\\ S \text{ contains a geometrically irreducible subscheme.}\end{array} \tag{$*$}$$

We have shown that $(*)$ holds for

$$\mathbb{S} = \{\text{degenerations of Fano varieties}\}.$$

There are two immediate generalizations, but (∗) fails for both. First, degenerations of Fano varieties are all rationally chain connected, that is, any two points are connected by a chain of rational curves. (See [Kol96, Chap.IV] for a general overview.)

The triangle $(xyz = 0) \subset \mathbb{P}^2$ is rationally chain connected. Let $K/\mathbb{Q}$ be any cubic extension with norm form $N(x, y, z)$. Then $C_N := (N(x, y, z) = 0)$ has no geometrically irreducible $\mathbb{Q}$-subvarieties but it is isomorphic to the triangle over $\bar{\mathbb{Q}}$.

One can also try to work with singular Fano schemes. That is, schemes X such that ω_X is a line bundle such that ω_X^{-1} is ample. Here (∗) again fails.

Take the affine variety $(N(x, y, z) + x^4 + y^4 + z^4 = 0) \subset \mathbb{A}^3$. Blow up the origin to get Y. The exceptional curve is isomorphic to C_N, let I be its ideal sheaf. Then $X = \operatorname{Spec}_Y \mathcal{O}_Y/I^2$ is a Fano scheme with no geometrically irreducible $\mathbb{Q}$-subvarieties.

I have, however, no counter examples to the following questions:

Question 10. Does (∗) hold for the following two classes of schemes:

(1) Degenerations of smooth rationally connected varieties.
(2) Reduced Fano schemes.

In fact, in both cases it may be true that such a scheme contains a geometrically irreducible subscheme which is also rationally connected. The following variant is especially interesting:

Question 11. Let k be a field of characteristic 0, C a smooth k-curve, Z a smooth k-variety and $g : Z \to C$ a projective morphism. Assume that

(1) the generic fiber F_{gen} is rationally connected,
(2) every fiber is a normal crossing divisor, and
(3) for every $c \in C$, every $k(c)$-irreducible component of $g^{-1}(c)$ is smooth.

Is it true that every fiber $g^{-1}(c)$ contains a $k(c)$-irreducible component which is rationally connected (and hence geometrically irreducible)?

Remark 12 (Positive characteristic)**.** The conjecture of Ax needs only minor modifications in positive characteristic, see [FJ05, Chap.21].

It is known that for any prime p, the following are equivalent:

(1) $\mathbb{F}_p(t)$ is weakly C_1.
(2) Every field of characteristic p is weakly C_1.
(3) Every perfect PAC field of characteristic p is C_1.

My proof has 2 difficulties in general. First, resolution of singularities is not known (but it is expected to be true). Second, Kodaira's vanishing theorem and its generalization (5) are false in positive characteristic. As far as I know, however, the Kollár–Shokurov Connectedness Theorem may hold in positive characteristic.

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